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Accuracy of Rational-Function Approximations for Linear Transonic Unsteady Aerodynamics

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Nomenclature

$[A_m]$ = coefficient matrices
b_n = poles, lag-parameters
$[C]$ = damping matrix
$[H]$ = oscillatory aerodynamic matrix
J = total squared fit-error
$[K]$ = stiffness matrix
$[M]$ = mass matrix
$[Q]$ = unsteady aerodynamic transfer-function matrix
s = Laplace variable
w_{ij} = weighting factor
ε_{ij} = squared fit-error of element, i, j
$\{\xi\}$ = generalized coordinates
ω = frequency of oscillation

Introduction

FOR the linear algebraic analysis of a time-invariant aeroservoelastic system, the unsteady aerodynamic forces are desired as rational-functions in the Laplace domain. Many approaches have been developed for arriving at such rational-function approximations (RFA), including those which involve optimization for the lag terms of the unsteady aerodynamic transfer-function matrix.[1–8] It was discovered recently that when the lag terms (or poles) of the aerodynamic transfer-function are optimized, repeated optimum lag-parameter values are obtained frequently, which require the use of new multiple-pole RFA.[7–9] The new approximations are consistent in that they never lead to the ill-conditioned state-space model of the conventional repeated simple-pole approximations. The RFA are fitted to linear frequency-domain aerodynamics and then extended to the entire Laplace domain by the method of analytic continuation. The assumption of linearity is valuable in a dynamic system analysis and is sought whenever possible. Although the transonic unsteady aerodynamics is governed by nonlinear field equations, Dowell et al.[10] indicated that it is possible in many cases to linearize the unsteady transonic airloads. This Note examines the accuracy of established RFA in the description of linearized transonic unsteady aerodynamics.

Rational-Function Approximations

The equation of motion of an aeroelastic system in Laplace domain can be written in terms of the generalized coordinates as

$$([M]s^2 + [C]s + [K])\{\xi(s)\} = [Q(s)]\{\xi(s)\} \tag{1}$$

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For a linear, causal, and stable system, $[Q(s)]$ is analytic and can be deduced from $[Q(i\omega)]$, which is obtained from an oscillatory theory, such as the doublet-lattice method[11] and the doublet-point method.[12] This process of analytic continuation is achieved by approximating each term of $[Q(s)]$ by a rational polynomial in s. The conventional simple-pole RFA, developed by Sevart,[1] Roger,[2] and Abel,[3] is expressed as

$$[Q(s)] = [A_0] + [A_1]s + [A_2]s^2 + \sum_{n=1}^{nL} \frac{[A_{(n+2)}]}{s + b_n} \tag{2}$$

The coefficients $[A_m]$ are determined by using a least-squares curve-fit of $[Q(i\omega)]$ with the tabulated frequency-domain data, $[H(\omega)]$, obtained from an oscillatory theory. The normalized squared error of this curve-fit is given by

$$\varepsilon_{ij} = \sum_{m=1}^{N_\omega} \frac{|Q_{ij}(i\omega_m) - H_{ij}(\omega_m)|^2}{R_{ij}(\omega_m)} \tag{3}$$

where

$$R_{ij}(\omega_m) = \max\{1, |H_{ij}(\omega_m)|^2\}$$

The least-squares fitting process translates into

$$\frac{\partial \varepsilon_{ij}}{\partial (A_m)_{ij}} = 0 \tag{4}$$

b_n in Eq. (2) are determined by a nonlinear optimization process, which minimizes the objective function

$$J = \sum_{m=1}^{N_\omega} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij}\varepsilon_{ij} \tag{5}$$

w_{ij} are taken to be unity in this study. Dunn[4] and Karpel[5] used gradient-based optimization for determining the lag-parameters, while Refs. 6–8 and 9 employed a Simplex non-gradient optimization procedure which is also used here. For a subsonic case, Eversman and Tewari[7] showed that when repeated optimum lag-parameter values occur, a new multiple-pole approximation is required for $[Q(s)]$, such as

$$[Q(s)] = [A_0] + [A_1]s + [A_2]s^2 + \sum_{n=1}^{N_1} \frac{[A_{(n+2)}]}{s + b_n} + \sum_{n=N_1+1}^{N_2} \frac{[A_{(n+2)}]}{\{s + b_n\}^2} + \cdots \tag{6}$$

where N_1 is the total number of poles, $(N_2 - N_1)$ is the number of poles repeated twice or more times, etc. Tewari[8] observed that the multiple-pole approximations are needed at supersonic speeds also. Tewari and Brink-Spalink[9] gave an analytical explanation for the requirement of multiple-pole RFA.

Rational-Function Approximations in the Transonic Regime

In order to examine the accuracy of RFA given by Eqs. (2) and (6) for the linearized transonic unsteady aerodynamics determined in frequency-domain by the doublet-lattice[11] and

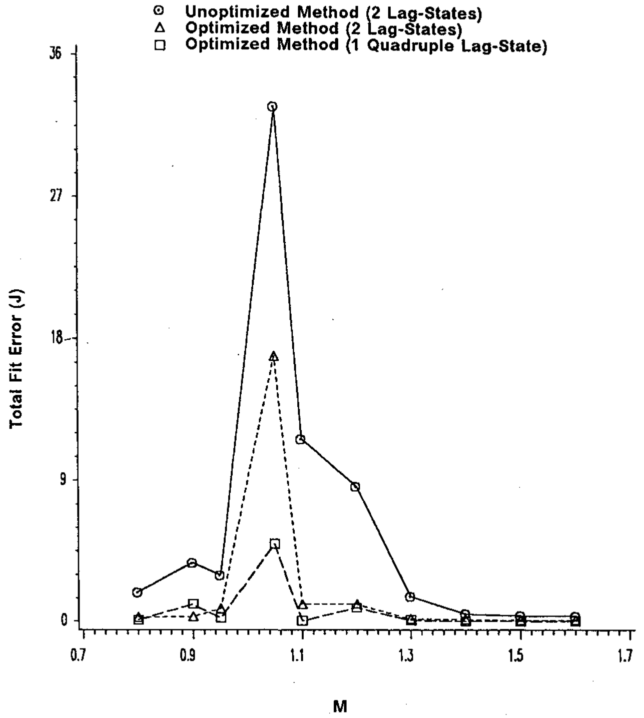

Fig. 1 Variation of the total fit-error with Mach number for two lag-states in the simple-pole RFA.

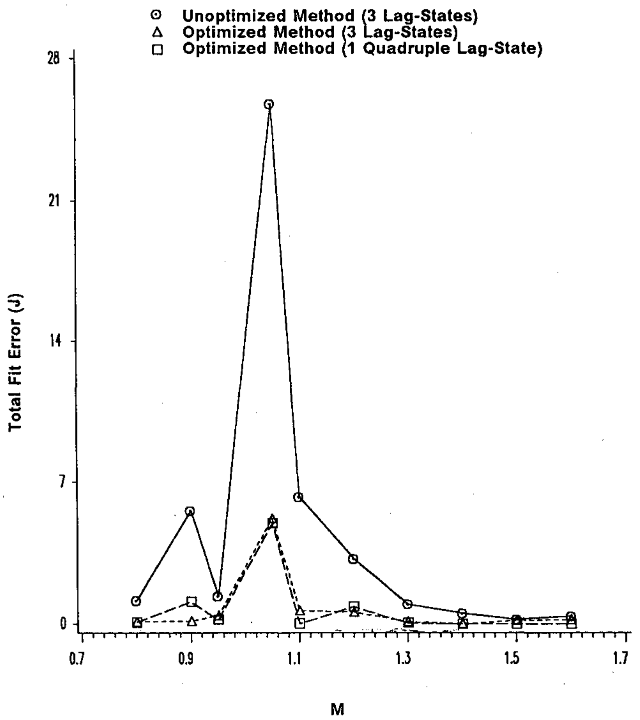

Fig. 2 Variation of the total fit-error with Mach number for three lag-states in the simple-pole RFA.

supersonic doublet-point[12] methods, consider the following numerical example. The planform geometry of the high aspect-ratio test wing, and the set of reduced-frequencies used for the fit are the same as in Ref. 7, but the wing is stiffened in order to have the flutter-speed in supersonic regime. Only the first six structural modes are retained.[9] Figure 1 presents J against the Mach number. The simple-pole RFA of Eq. (2) with two poles, and the multiple-pole RFA with a quadruple pole [Eq. (6)] are considered. The fit-error for unoptimized simple-pole RFA is seen to increase dramatically in the Mach number range 0.9–1.2. While the fit-error is brought down

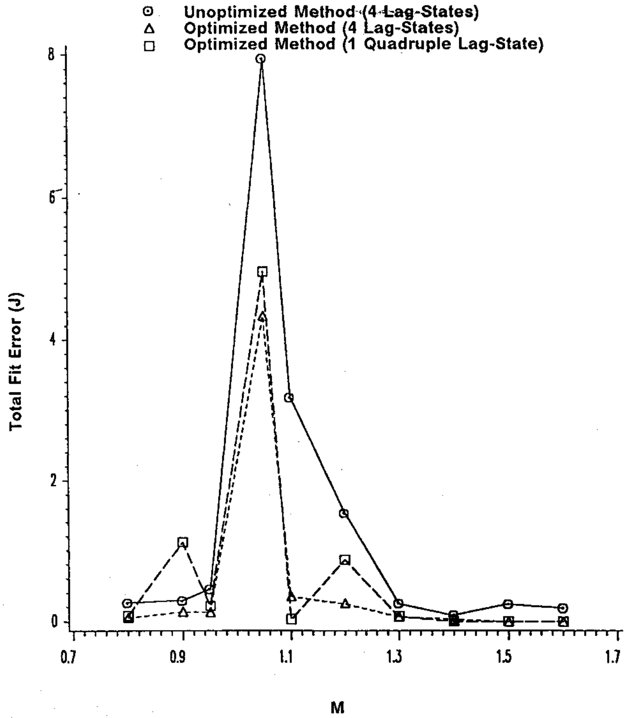

Fig. 3 Variation of the total fit-error with Mach number for four lag-states in the simple-pole RFA.

considerably by optimization at $M = 0.9$, 0.95, and 1.1, the optimized fit-error remains high at $M = 1.05$. A similar tendency is observed in Figs. 2 and 3, which result from 3 and 4 simple poles, respectively, in the RFA of Eq. (2). These figures indicate that the RFA are less accurate in the transonic range than at subsonic or supersonic regimes. For a better fit-accuracy in the transonic regime, these RFA require an increased number of lag-states, thereby increasing the order of the state-space model. It may be advisable to investigate suitable weighting factors for the lag terms for an improvement in the fit-accuracy, since it appears that the transonic fit is more sensitive to the manner in which the lag terms are expressed, when compared to the fit at other Mach numbers.

Conclusion

The fit-accuracy of traditional simple-pole, and new multiple-pole, optimized rational-function approximations fitted to linearized frequency-domain aerodynamic data, degrades considerably in the transonic range. New approximations should be sought for the lag terms of the unsteady aerodynamic transfer-function in order to have a transonic fit-accuracy comparable to that at other Mach numbers.

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Fractional Order Calculus Model of the Generalized Theodorsen Function

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Introduction

THE generalized Theodorsen function relates unsteady, circulatory lift to unsteady circulation for small perturbation motion of a flat plate in two-dimensional, incompressible, inviscid flow.[1] Because the generalized Theodorsen function is based on first principles, it is an appropriate bench mark for modeling unsteady aerodynamic forces on thin airfoils. The generalized Theodorsen function is a Laplace transform composed of tanscendental functions and it has no closed-form inverse transform. These characteristics make the generalized Theodorsen function somewhat cumbersome to manipulate in analyses. This motivates the development of algebraic approximations, having inverse transforms, that are suitable for use in aeroelastic stability analyses, gust response predictions, and control system design.

It is possible to construct an algebraic, global s-plane approximation of the generalized Theodorsen function using fractional powers of s. The approximation is mathematically compact and has a known, but unfamiliar, inverse transform that is also compact. The approximation also leads to equations of motion equally well-posed in either the Laplace domain or the time domain.[2]

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Background

The generalized Theodorsen function $C(s)$ is a combination of Hankel functions of the first kind, H_0 and H_1:

$$C(s) = \frac{H_1(\bar{s})}{H_0(\bar{s}) + H_1(\bar{s})}, \qquad \bar{s} = \frac{sb}{U} \tag{1}$$

Here, $\bar{s}$ is a dimensionless form of the Laplace parameter, where b is the flat plate airfoil's semichord, and U is the freestream velocity. The Theodorsen function relates the plate's circulatory lift to the circulation of the flow about the plate

$$L(s) = \rho U \Gamma(s) C(s) \tag{2}$$

where $L(s)$ and $\Gamma(s)$ are the Laplace transforms of the time-dependent circulatory lift and the time-dependent circulation, respectively.

Unfortunately, the Theodorsen function has no known, closed-form representation of its inverse transform. This difficulty, coupled with the transcendental nature of the Hankel functions, has motivated the formulation of algebraic models of Theodorsen's function.

Commonly encountered algebraic models take the form

$$C(s) \approx \frac{N(s)}{D(s)} = \frac{\sum_{n=0}^{N} a_n \bar{s}^n}{\sum_{n=0}^{N} b_n \bar{s}^n} \tag{3}$$

where $N(s)$ and $D(s)$ are polynomial functions of the Laplace parameter $\bar{s}$. These models have inverse transforms that are a unit impulse function plus a combination of elementary analytic functions, typically exponentials.

A drawback of these models is their poles that arise from the roots of $D(s)$. Theodorsen's function has no poles in the s plane. Consequently, the poles in the models are sources of substantial, but localized, inaccuracies that detract from the model's fidelity. When one of these models is used to describe aerodynamic forces in equations of motion for an airfoil, attention must be devoted to keeping the poles introduced by the model well away from poles associated with airfoil motion. As a result, different models might be needed for flutter analysis, divergence analysis, and control system design, respectively.

Fractional Calculus Model

The primary motive for introducing concepts from the fractional order calculus is to construct a single, high-fidelity, global s-plane model of Theodorsen's function suitable for all the above analyses. This new model employs the notion of differentiating to fractional order. The extended Riemann-Liouville definition[3] for the fractional β order derivative of the function $x(t)$ is

$$D^{\beta}[x(t)] \equiv \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{x(t-\tau)}{\tau^{\beta}} \, \mathrm{d}\tau \qquad 0 < \beta < 1 \tag{4}$$

Where Γ denotes the gamma function instead of circulation. The Laplace transform of Eq. (4) reveals the operative property of fractional order differentiation:

$$\mathcal{L}\{D^{\beta}[x(t)]\} = s^{\beta}\mathcal{L}\{x(t)\} \tag{5}$$

The fractional calculus model of Theodorsen's function, $\hat{C}(s)$, shown here, contains fractional powers of $\bar{s}$:

$$\hat{C}(s) = \frac{1 + (2.19)\bar{s}^{5/6}}{1 + 2(2.19)\bar{s}^{5/6}}, \qquad \bar{s} = \frac{sb}{U} \tag{6}$$

This model is based on a visual "fitting" of the model to the function along the imaginary $\bar{s}$ axis. This model is compared